Isocurvature Perturbations in Quintessence Cosmologies

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We present a systematic treatment of the initial conditions and evolution of cosmological perturbations in a universe containing photons, baryons, neutrinos, cold dark matter, and a scalar quintessence field. By formulating the evolution in terms of a differential equation involving a matrix acting on a vector comprised of the perturbation variables, we can use the familiar language of eigenvalues and eigenvectors. As the largest eigenvalue of the evolution matrix is fourfold degenerate, it follows that there are four dominant modes with non-diverging gravitational potential at early times, corresponding to adiabatic, cold dark matter isocurvature, baryon isocurvature and neutrino isocurvature perturbations. We conclude that quintessence does not lead to an additional independent mode.

Introduction

We present here a systematic treatment of initial conditions for quintessence models. The universe we will consider contains photons, baryons, neutrinos, cold dark matter and a scalar quintessence field. We will formulate the evolution equations for the perturbation variables as a first order differential matrix equation:

$$\frac{\mathrm{d}}{\mathrm{d}\ln x}\boldsymbol{U} = A(x)\boldsymbol{U}, \quad (1)$$

where the vector $\boldsymbol{U}$ contains all perturbation variables and the matrix $A(x)$ encodes the evolution equations. In doing so, we relate the problem of finding initial conditions and dominant modes to the familiar language of eigenvalues and eigenvectors. This formulation makes "mode-accounting" transparent by counting the degeneracy of the largest eigenvalue. We find four dominant modes that remain regular at early times. For physical reasons, we choose a basis using adiabatic, CDM isocurvature, baryon isocurvature and neutrino isocurvature initial conditions.

The Different Modes

In the following we adopt the gauge-invariant approach as devised by Bardeen [1]. For a more detailed derivation of the perturbation equations see Doran et al.[2]. It turns out that the evolution is best described as a function of $x \equiv k\tau$, where τ is the conformal time and k the comoving wavenumber of the mode. We assume that at early times, the universe expands as if radiation dominated. Assuming tracking quintessence we obtain the following set of equations :

$$\Delta_c' = -x^2\tilde{V}_c \quad (2) \qquad \tilde{V}_c' = -2\tilde{V}_c + \Psi \quad (3) \qquad \Delta_\gamma' = -\frac{4}{3}x^2\tilde{V}_\gamma \quad (4)$$

$$\tilde{V}_\gamma' = \frac{1}{4}\Delta_\gamma - \tilde{V}_\gamma + \Omega_\nu \tilde{\Pi}_\nu + 2\Psi \quad (5) \qquad \Delta_b' = -x^2 \tilde{V}_\gamma \quad (6) \qquad \Delta_\nu' = -\frac{4}{3}x^2 \tilde{V}_\nu \quad (7)$$

$$\tilde{V}_\nu' = \frac{1}{4}\Delta_\nu - \tilde{V}_\nu - \frac{1}{6}x^2 \tilde{\Pi}_\nu + \Omega_\nu \tilde{\Pi}_\nu + 2\Psi \quad (8) \qquad \tilde{\Pi}_\nu' = \frac{8}{5}\tilde{V}_\nu - 2\tilde{\Pi}_\nu \quad (9)$$

$$\Delta_q' = 3(w_q - 1)\left[\Delta_q + 3(1+w_q)\left\{\Psi + \Omega_\nu \tilde{\Pi}_\nu\right\} + \left\{3 - \frac{x^2}{3(w_q-1)}\right\}(1+w_q)\tilde{V}_q\right] \tag{10}$$

$$\tilde{V}_q' = 3\Omega_\nu \tilde{\Pi}_\nu + \frac{\Delta_q}{1+w_q} + \tilde{V}_q + 4\Psi \quad (11) \qquad \Psi = -\frac{\sum_{\alpha=c,b,\gamma,\nu,q} \Omega_\alpha(\Delta_\alpha + 3(1+w_\alpha)\tilde{V}_\alpha)}{\sum_{\alpha=c,b,\gamma,\nu,q} 3(1+w_\alpha)\Omega_\alpha + \frac{2x^2}{3}} - \Omega_\nu \tilde{\Pi}_\nu \quad (12)$$

with the gauge-invariant Newtonian potential Ψ. We denote the derivative $\mathrm{d}/\mathrm{d}\ln x$ with a prime. The gauge-invariant energy density contrasts Δ_α, the velocities $\tilde{V}_\alpha$ and the shear $\tilde{\Pi}_\nu$ are the ones found in the literature [1,3,4], except that we factor out powers of x from the velocity and shear defining $\tilde{V} \equiv V/x$ and $\tilde{\Pi}_\nu \equiv x^{-2}\Pi_\nu$. The index α runs over the five species in our equations, quintessence is assigned the subscript q. We assume tight coupling between photons and baryons. The equation of state $w = \bar{p}/\bar{\rho}$ takes on the values $w_c = w_b = 0$, $w_\gamma = w_\nu = 1/3$ and w_q is left as a free parameter.

We define the perturbation vector as $\quad \boldsymbol{U}^T \equiv (\Delta_c, \tilde{V}_c, \Delta_\gamma, \tilde{V}_\gamma, \Delta_b, \Delta_\nu, \tilde{V}_\nu, \tilde{\Pi}_\nu, \Delta_q, \tilde{V}_q). \quad (13)$

The matrix $A(x)$ can easily be read off from equations (2)-(11). This enables us to discuss the problem of specifying initial conditions in a systematic way.

The initial conditions are specified for modes well outside the horizon, i.e. $x \ll 1$. In this case, the r.h.s. of equations (2), (4), (6) and (7) can be neglected, provided $\tilde{V}_\alpha$ does not diverge $\propto x^{-2}$ or faster for $x^2 \to 0$.

The general solution to Equation (1) in the (ideal) case of a truly constant A would be

$$\boldsymbol{U}(x) = \sum_i c_i \left(\frac{x}{x_0}\right)^{\lambda_i} \boldsymbol{U}^{(i)}, \tag{14}$$

where $\boldsymbol{U}^{(i)}$ are the eigenvectors of A with eigenvalue λ_i and the time independent coefficients c_i specify the initial contribution of $\boldsymbol{U}^{(i)}$ towards a general perturbation $\boldsymbol{U}$. As time progresses, components corresponding to the largest eigenvalues λ_i will dominate. Compared to these "dominant" modes, initial contributions in the direction of eigenvectors $\boldsymbol{U}^{(i)}$ with smaller $\mathrm{Re}(\lambda_i)$ decay. In our case, the characteristic polynomial of $A(x)$ indeed has a fourfold degenerate eigenvalue $\lambda = 0$ in the limit $x^2 \to 0$, the other six remaining eigenvalues are negative. We therefore need to solve $A(x)\mathbf{U} = \mathbf{0}$ which is equivalent to setting the l.h.s. of Equations (2)-(11) equal to zero and using $\Omega_c = \Omega_b = x^2 = 0$. Then Equations (2), (4), (6) and (7) are automatically satisfied (provided $\tilde{V}_\alpha$ does not diverge $\propto x^{-2}$ or faster), and Equations (3),(5),(8)-(11) yield non-trivial constraints for the components of $\boldsymbol{U}$:

$$2\tilde{V}_c - \Psi = 0 \quad (15) \qquad 1/4\Delta_\gamma - \tilde{V}_\gamma + \Omega_\nu \tilde{\Pi}_\nu + 2\Psi = 0 \quad (16)$$

$$1/4\Delta_\nu - \tilde{V}_\nu + \Omega_\nu\tilde{\Pi}_\nu + 2\Psi = 0 \quad (17)$$

$$8/5\tilde{V}_\nu - 2\tilde{\Pi}_\nu = 0 \quad (18)$$

$$3\Omega_\nu\tilde{\Pi}_\nu + \Delta_q/(1+w_q) + 3\tilde{V}_q + 3\Psi = 0 \quad (19)$$

$$3\Omega_\nu\tilde{\Pi}_\nu + \Delta_q/(1+w_q) + \tilde{V}_q + 4\Psi = 0 \quad (20)$$

Following the existing literature, we use the gauge-invariant entropy perturbation [3] between two species α and β, as well as the gauge-invariant curvature perturbation on hyper-surfaces of uniform energy density of species α in order to classify the physical modes[5,6,7,8] :

$$S_{\alpha:\beta} = \frac{\Delta_\alpha}{1+w_\alpha} - \frac{\Delta_\beta}{1+w_\beta}, \quad (21)$$

$$\zeta_\alpha = \left(H_L + \frac{1}{3}H_T\right) + \frac{\delta\rho_\alpha}{3(1+w_\alpha)\bar{\rho}_\alpha}. \quad (22)$$

In our variables, these expressions take on the manifestly gauge-invariant form

$$\zeta_\alpha = \frac{\Delta_\alpha}{3(1+w_\alpha)}\ , \quad \zeta = \frac{\sum_\alpha \Delta_\alpha\Omega_\alpha}{\sum_\alpha 3(1+w_\alpha)\Omega_\alpha}. \quad (23)$$

The first (rather intuitive) perturbations one would try to find are adiabatic perturbations, which are specified by the adiabaticity conditions $S_{\alpha:\beta} = 0$ for all pairs of components, i.e.

$$\Delta_\nu = \Delta_\gamma = \frac{4}{3}\Delta_c = \frac{4}{3}\Delta_b, \quad (24)$$

Using the six constraint Equations (15)-(20), we obtain the adiabatic mode. (Due to limited lenght of this article we cannot quote the full results, we therefore refer the reader to Doran et. al. [2] for more details.) We also conclude that quintessence is automatically adiabatic if CDM, baryons, neutrinos and radiation are adiabatic, independent of the quintessence model for as long as we are in the tracking regime.

Let us next consider the neutrino isocurvature mode. For this, we require that CDM, baryons and radiation are adiabatic, while $S_{\nu:\gamma} \neq 0$ and that the gauge-invariant curvature perturbation vanishes:

$$\zeta = 0, \quad \Delta_c = \Delta_b = \frac{3}{4}\Delta_\gamma. \quad (25)$$

Using this and Equations (15)-(20) leads to the neutrino-isocurvature perturbation.

It is important to note that we did not require quintessence to be adiabatic. One can see from the neutrino isocurvature vector that $\Delta_q = 0$, and as a consequence quintessence is not adiabatic with respect to either neutrinos, radiation, baryons or CDM. Hence, we could just as well have labeled this vector “quintessence isocurvature”.

The CDM isocurvature mode is characterized by $S_{c:\gamma} \neq 0$, $\zeta = 0$ and adiabaticity between photons, neutrinos and baryons. Similarly, for the baryon isocurvature mode we require $S_{b:\gamma} \neq 0$, $\zeta = 0$.

The adiabatic, CDM isocurvature, baryon isocurvature and neutrino isocurvature- vector are linearly independent. We have therefore identified four modes corresponding to the fourfold degenerate eigenvalue zero of $A(x)$. These four vectors span the subspace of dominant modes in the super-horizon limit. Arbitrary initial perturbations may therefore be represented by projecting a perturbation vector $\boldsymbol{U}$ at initial time into the subspace spanned by the four aforementioned vectors, as this is the part of the initial perturbations which will dominate as time progresses.

We use a modified version of CMBEASY [10,11] to compute CMB spectra corresponding to different initial conditions for an early quintessence cosmology with parameters as in model A of [12]. We set the spectral index of the isocurvature modes identical to the spectral index of the pure adiabatic mode, $n_s = 0.99$. Comparison with the WMAP data in the same figure shows that non-adiabatic initial perturbations are strongly constrained.

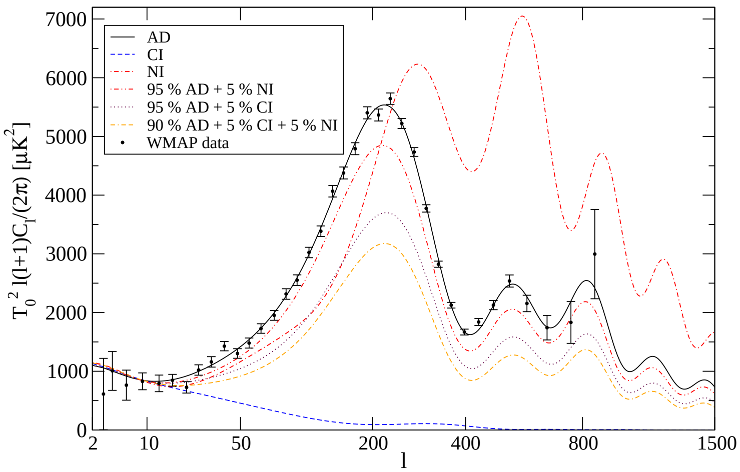

Figure 1: CMB Temperature spectra as a function of multipole l in an early quintessence cosmology. The pure adiabatic (AD), CDM isocurvature (CI), neutrino isocurvature (NI) mode and three different combinations of these dominant modes are plotted. For comparison with experimental data we also give the WMAP measurements of the CMB[9]. The spectrum of the pure baryon isocurvature mode is essentially identical to that of the pure CDM isocurvature mode. All spectra have been normalized to the same power at $l = 10$

Conclusion

We have investigated perturbations in a radiation-dominated universe containing quintessence, CDM, neutrinos, radiation and baryons in the tight coupling limit. The perturbation evolution has been expressed as a differential equation involving a matrix acting on a vector comprised of the perturbation variables. This formulation leads to a systematic determination of the initial conditions. In particular, we find that due to the presence of tracking scalar quintessence no additional dominant mode is introduced. This fact is beautifully transparent in the matrix language. In total, we find four dominant modes and choose them as adiabatic, CDM isocurvature, baryon isocurvature and neutrino isocurvature. For the neutrino isocurvature mode, quintessence automatically is forced to non-adiabaticity. Hence, we could have as well labeled the neutrino isocurvature mode as quintessence isocurvature. To demonstrate the influence on the cosmic microwave background anisotropy spectrum, we have calculated spectra for all modes. A detailed study may provide ways to put additional constraints on quintessence models or tell us more about the initial perturbations after inflation.

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